A

PROJECT REPORT ON

"CONTINUED FRACTION"

HKES'S A V PATIL ARTS SCIENCE & COMMERCE COLLEGE ALAND"

Submitted to the

H.K.E. Society's

A V PATIL ARTS SCIENCE & COMMERCE COLLEGE ALAND"

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CERTIFICATE OF COMPLETION

This is to certify that the following students have successfully completed the project on **"CONTINUED FRACTION"** AT HKES'S A V PATIL ARTS SCIENCE & COMMERCE COLLEGE ALAND" is based on the project carried out under the guidance of Laxman Rathod Associate Professor and is submitted to the Department of Science, H.K.E. Society's A V Patil Arts, Science & Commerce College Aland.

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CHAPTER 1

EXPANSION OF RATIONAL NUMBER

INTRODUCTION:

$$X = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + a_{n-1} + \cfrac{1}{a_n}}}}$$

Such a multilayered function is a continued fraction.

A term coined by the English Mathematician John Wallis (1616-1703). His book, Opera Mathematica (1695) contains some basic work on continued fractions.

Aryabhata used them to solve specific LDE"s. Italian Mathematician Rafael Bombelli (1526-1573) is often credited with laying the foundation for the theory of continued fractions.

Although these Mathematicians made contributions to the development of continued fractions, the modern theory of such fractions did not flourish until Euler John Heinrich Lambert (1728-1777) and Langrage embraced the topic. Euler studied them around 1730 and his De Fractionlous Continuous (1737) contains much of work. In1759, he employed them to solve equation of form $x^2 - Ny^2 = 1$, called Pell"s equation. Seven years later Langrange developed the fundamental properties of periodic continued fractions.

Continued fraction was developed by the Indian Mathematical genius Srinivasa Ramanujan, who studied them in 1908.

SRINIVASA RAMANUJAN

(1887-1920)

The great Indian Mathematician was born on Dec 22, 1887 in Erode, near Madras, the son of a bookkeeper at a cloth store in Kumbakonam. After two years of elementary school he transferred to the high school at age seven. At ten he placed first in the district primary examination.

In 1903, his passion for Mathematics was sparked when he borrowed a copy of George Schoobridge Carr's A Synopsis of elementary results in pure and applied Mathematics from a University student. Without any formal training or outside help Ramanujan established the 6000 theorems in the book, stated without proofs or any explanation and kept their results in a notebook.

Ramanujan arrived in Cambridge in 1914 with the help of a scholarship arranged by English Mathematician Godfrey H. Hardy of Cambridge University. During his five year stay, he and Hardy collaborated on a number of articles in the theory of partitions, analytic number theory, continued fractions, infinite series, and elliptic functions.

In 1917, Ramanujan became seriously ill. He was incorrectly diagnosed with tuberculosis; however, it is now believed that he suffered from a vitamin deficiency caused by his strict vegetarianism.

When Ramanujan was sick in a nursing home, Hardy visited him. Hardy told him that the number of the cab he came in, 1729, was a "rather dull number" and he hoped that it wasn"t a bad omen. "No, sir," Ramanujan responded. "It is a very interesting number. It is the smallest no expressible as the sum of two cubes in two different ways."

In 1918, Ramanujan became one of the youngest members of the Fellow of the Royal Society and a fellow of Trinity College.

Ramanujan returned to India the following year. He pursued his mathematical passion even on his deathbed. His short but extremely productive life ended when he was only 32.

Definitions and Notation

An expression of the form

$$X = a_0 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 \ldots\ldots}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is called a continued fraction. In general, the numbers all $a_1, a_2, \ldots, b_1, b_2, b_3, \ldots$ may be any real or complex numbers, and the number of terms may be finite or infinite.

In this monograph, however, we shall restrict our discussion to simple continued fractions. These have the form

$$X = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where the first term a_1 is usually a positive or negative integer (but could be zero), and where the terms, $a_2, a_3, a_4 \ldots.$ are positive integers. In fact, These have the form,

$$X = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 \ldots\ldots\ldots + \cfrac{1}{a_n + \cfrac{1}{a_n}}}}}} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

With only a finite number of terms $a_1.a_2, a_3, \ldots, a_n$ Such a fraction is called a terminating continued fraction. From now on, unless the contrary is stated, the words continued fraction will imply that we are dealing with a finite simple continued fraction. A much more convenient way of writing (3) is

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \cdots + \frac{1}{a_n} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

Where the + signs after the first one are lowered to remind us of the "step-down" process in forming a continued fraction. It is also convenient to denote the continued fraction (4) by the symbol $[a_1, a_2, \ldots, a_n]$, so that

$$[a_1, a_2, \ldots, a_n] = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} \qquad \ldots\ldots\ldots\ldots(5)$$

The terms $a_1, a_2, \ldots, a_n$ are called the partial quotients of the continued fraction.

Expansion of Rational Fractions

A rational number is a fraction of the form p/q where p and q are integers with $q \neq 0$. We shall prove in the next section that every rational fraction, or rational number, can be expressed as a finite simple continued fraction.

For example, the continued fraction for 67/29 is

$$\frac{67}{29} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}} = 2 + \frac{1}{3} + \frac{1}{4} + \frac{1}{2}$$

$$\frac{67}{29} = [2,3,4,2]$$

How did we get this result? First we divided 67 by 29 to obtain the quotient 2 and the remainder 9, so that

$$\frac{67}{29} = 2 + \frac{9}{29} = 2 + \cfrac{1}{\frac{29}{9}} \qquad \ldots\ldots\ldots\ldots(6)$$

Note that on the right we have replaced 9/29 by the reciprocal of 29/9 Next we divide 29 by 9 to obtain

$$\frac{29}{9} = 3 + \frac{2}{9} = 3 + \cfrac{1}{\frac{9}{2}} \qquad \ldots\ldots\ldots\ldots(7)$$

Finally, we divided 9 by 2 to obtain

$\frac{9}{2} = 4 + \frac{1}{2}$ ....(8)

at which stage the process terminates. Now substitute (8) into (7), and then substitute (7) into (6) to get

$$\frac{67}{29} = 2 + \frac{1}{29/9} = 2 + \frac{1}{3 + \frac{1}{\frac{9}{2}}} = 2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}}$$

$\frac{67}{29} = [2,3,4,2] = [a_1, a_2, a_3, a_4]$ ....(9)

We should notice that in equation (7) the number 2 .29 is the largest multiple of 29 that is less than 67, and consequently the remainder (in this case the number 9) is necessarily a number>0.

but definitely < 29.

Next consider equation (7). Here 3.9 is the largest multiple of 9 that is less than 29. The remainder, 2, is necessarily a number ≥0 but < 9.

In (8) the number 4.2 is the largest multiple of 2 that is less than 9 and the remainder is 1, a number ≥0 but < 2. Finally, we cannot go beyond equation (8), for if we write,

$$\frac{9}{2} = 4 + \frac{1}{\frac{2}{1}} = 4 + \frac{1}{2}$$

then 2.1 is the largest multiple of 1 that divides 2 and we simply end up with

$$\frac{2}{1} = 2.1 + 0 = 2$$

so the calculation terminates.

If a number a is less than a number b we write $a < b$. If a is less than or qua1 to b we write $a \leq b$. Likewise, if a is greater than b, or if a is greater ,k' than or equal to b, we write, respectively, $a > b$, $a \leq b$. For a detailed discussion of inequalities, see E. Beckenbach and R. Bellman.

The process for finding the continued fraction expansion for 67/29 can be arranged as follows:

29) 67 (2 = a_1	Divide 67 by 29.
58	2 .29 = 58; subtract 58 from 67.
9)29(3 = a_2	Divide 29 by 9.
27	3 9 = 27; subtract 27 from 29.
2)9(4 = a_3	Divide 9 by 2.
8	4 .2 = 8; subtract 8 from 9.
1)2(2 = a_4	Divide 2 by 1.
	2 2 1 = 2; subtract 2 from 2. 0 Process terminates.

Thus 67/29=[2,2,4,2]=[a_1,a_2,a_3,a_4]

We observe, in this example, that in the successive divisions the remainders 9, 2, 1 are exactly determined non-negative numbers each smaller than the corresponding divisor. Thus the remainder 9 is less than the divisor 29, the remainder 2 is less than the divisor 9, and so on. The remainder in each division becomes the divisor in the next division, so that the successive remainders become smaller and smaller non-negative integers. Thus the remainder zero must be reached eventually, and the process must end.

Each remainder obtained in this process is a unique non-negative number. For example, can you divide 67 by 29, obtain the largest quotient 2, and end up with a remainder other than 9? This means that, for the given fraction 67/29, our process yields exactly one sequence of remainders.

Ex:2) As a second example, let us find the continued fraction expansion for

29/67 we obtain.

$67\overline{)29}(0 = a_1$

0

$29\overline{)67}(2 = a_2$

58

$9\overline{)29}(3 = a_3$

27

$2\overline{)9}(4 = a_4$

8

$1\overline{)2}(2 = a_5$

2

$\overline{0}$

$\frac{29}{67} = [0,2,3,4,2] = [a_1, a_2, a_3, a_4, a_5,]$.

Notice that in this example $a_1 = 0$. To check our results, all we have to do is simplify the continued fraction

$$0 + \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\frac{1}{2}}}} = \cfrac{1}{2+\cfrac{1}{3+\frac{2}{9}}} = \cfrac{1}{2+\frac{9}{29}} = \frac{29}{67}$$

A comparison of the expansion 67/29 = [2,3,4,2] with the expansion of the reciprocal 29/67 =[0, 2, 3,4, 2] suggests the result that, if p is greater than q and then

$\frac{p}{q} = [a_1, a_2, \ldots a_n]$

Then $\frac{q}{p} = [a, a_1, a_2, \ldots a_n]$

The reader is asked to state a similar result for p < q.

THEOREM 1: Any finite continued fraction represents a rational number.

Proof:

(We shall apply induction on the number of partial quotients.)

Let $[a_0; a_1, a_2, a_3, \ldots, a_n]$

be a finite simple continued fraction. When $n = 1$,

$$[a_0; a1] = a_0 + \frac{1}{a_1} = \frac{a_0\, a_1 + 1}{a_1} \text{ is a rational number.}$$

Now assume that every finite simple continued fraction with k partial quotients is a rational number, where $k \geq 1$. Then

$$[a0, a_1, a_2, a_3 \ldots, a_k, a_{k+1}] = a_0 + \frac{1}{[a_1, a_2 a_3, \ldots, a_k, a_{k+1}]}$$

Since $[a1, a2,3, \ldots, ak, ak{+}1]$ contains k partial quotients it is a rational number $\frac{r}{s}$, Where $s \neq 0$.

Then,

$$[a_0, a_1, a_2, a_3, \ldots, a_k, a_{k+1}] = a_0 + \frac{1}{r/s} = a_0 + \frac{s}{r} = \frac{a_0 r + s}{s}$$

is a rational number.

Thus, by induction $[a_0, a_1, a_2, a_3, \ldots, a_n]$ is a rational number for every positive integer n. This completes the proof.

EXAMPLE ON THEOREM 1:

1. The finite simple continued fraction is 87/49 = $1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{1}{5}}}}$

Solution: $2 + \frac{1}{5} = \frac{11}{5}$ $\qquad$ $3 + \frac{1}{\frac{11}{5}} = 3 + \frac{5}{11} = \frac{38}{11}$

$1 + \frac{1}{\frac{38}{11}} = 1 + \frac{11}{38} = \frac{49}{38}$ $\qquad$ $1 + \frac{1}{\frac{49}{38}} = 1 + \frac{38}{49} = \frac{87}{49}$

THEOREM 2: Every rational number can be represented by a finite simple continued fraction

Proof:

Let $x = a/b$ be a rational number where $b > 0$. For convenience, we get $r_0 = a$ and $r_1 = b$

By Euclidean algorithm we have

$r_0 = r_1 q_1 + r_2 \qquad 0 < r_2 < r_1$

$r_1 = r_2 q_2 + r_3 \qquad 0 < r_3 < r_2$

$r2 = r_3 q_3 + r_4 \qquad 0 < r_4 < r_3$

.

.

$r_{n-2} = r_{n-1}\, q_{n-1} + r_n \qquad 0 < r_n < r_{n-1}$

Where $q_1, q_2, \ldots, q_n$ are quotients, $r_1, r_2, \ldots, r_n$ are remainders and are positive.

It follows from the equations that

$$\frac{a}{b}=\frac{r_0}{r_1}=q_1+\frac{r_2}{r_1}=q_1+\frac{1}{r_1/r_2}$$

$$\frac{r_1}{r_2}=q_2+\frac{r_3}{r_2}=q_2+\frac{1}{\frac{r_2}{r_3}}$$

$$\frac{r_2}{r_3}=q_3+\frac{r_4}{r_3}=q_3+\frac{1}{r_3/r_4}$$

.

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$$\frac{r_{n-2}}{r_{n-1}}=q_{n-1}+\frac{r_n}{r_{n-1}}=q_{n-1}+\frac{1}{\frac{r_{n-1}}{r_n}}$$

$$\frac{r_{n-1}}{r_n}=q_n$$

Substituting for $\frac{r_1}{r_2}$ in the first equation yields

$$\frac{a}{b}=\frac{r_0}{r_1}=q_1+\frac{1}{\frac{r_1}{r_2}}=q_1+\frac{1}{q_2+\frac{r_3}{r_2}}$$

Continuing like this we get

$$\frac{a}{b} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_n}}}$$

$=[q_1, q_2, q_3, \ldots\ldots\ldots\ldots q_{n-1}, q_n]$

Thus, every rational number can be represented by a finite simple continued fraction.

EXAMPLE ON THEOREM 2:

1. Express in to finite simple continued fraction $\frac{89}{49}$

Solution: The Euclidean algorithm is given as $a = b \times q + r \quad 0 \le r < b$

$89 = 1 \times 49 + 40$

$49 = 1 \times 40 + 9$

$40 = 4 \times 9 + 4$

$9 = 2 \times 4 + 1$

$4 = 4 \times 1 + 0$

Thus,

$$\frac{89}{49} = 1 + \frac{40}{49} = 1 + \cfrac{1}{\frac{49}{40}}$$

$$\frac{49}{40} = 1 + \frac{9}{40} = 1 + \cfrac{1}{\frac{40}{9}}$$

$$\frac{40}{9} = 4 + \frac{4}{9} = 4 + \cfrac{1}{\frac{9}{4}}$$

$$\frac{9}{4} = 2 + \frac{1}{4} = 2 + \frac{1}{4}$$

we can express it as

$$\frac{89}{49} = 1 + \cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{2+\frac{1}{4}}}}$$

$$\frac{89}{49} = [1; 1, 4, 2, 4]$$

EXAMPLE: 2 Express 227/160 as finite simple continued fraction by using the Euclidean algorithm.

Solution: The Euclidean algorithm is given as $a = b \times q + r \quad 0 \leq r < b$

227 = 2 × 155 – 83

115 = 2 × 83 – 11

83 = 8 × 11 – 5

11 = 3× 5 – 4

5 = 2 × 4 –3

4 = 2 × 3 – 2

3 = 2 × 2 – 1

2 = 2 × 1 – 0

Thus,

$$\frac{227}{115} = 2 - \frac{83}{115} = 2 - \frac{1}{\frac{115}{83}}$$

$$\frac{115}{83} = 2 - \frac{11}{83} = 2 - \frac{1}{83/11}$$

$$\frac{83}{11} = 8 - \frac{5}{11} = 8 - \frac{1}{11/5}$$

$$\frac{11}{5} = 3 - \frac{4}{5} = 3 - \frac{1}{5/4}$$

$$\frac{5}{4} = 2 - \frac{3}{4} = 2 - \frac{1}{4/3}$$

$$\frac{4}{3} = 2 - \frac{2}{3} = 2 - \frac{1}{3/2}$$

$$\frac{3}{2} = 2 - \frac{1}{2} = 2 - \frac{1}{2/1}$$

$$\frac{2}{1} = 2 - 0$$

We can express it as

$$\frac{227}{155} = 2 - \cfrac{1}{2 - \cfrac{1}{8 - \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \frac{1}{2}}}}}}}$$

$$\frac{227}{115} = [2; 2, 8, 3, 2, 2, 2, 2]$$

CONVERGENCE OF CONTINUED FRACTIONS

The continued fraction for x = [a_0,a_1,$a_{2,}$, ... , a_n] at the various plus signs, we can generate a sequence {ck } of the approximation of x, where $0 \le k \le n$; thus, ck =[a_0, a_1, a_2, a_3, ... , a_k] ck is called the k^{th} convergent of x, a concept introduced by Wallis in his Opera Mathematics.

For example, you may verify that

$\frac{F_8}{F_7} = \frac{21}{13}$ = [1; 1, 1 , 1, 1, 1, 1]

The various convergent are

C_0 = [1] $=\frac{1}{1} = 1$

C_1 = [1; 1] $=\frac{2}{1} = 2$

C_2 = [1; 1, 1] $=\frac{3}{2} = 1.5$

C_3 = [1; 1, 1, 1] $=\frac{5}{3} = 1.6666667$

C_4 =]1; 1, 1, 1, 1] $=\frac{8}{5} =1.6$

C_5 = [1;1, 1, 1, 1, 1] $=\frac{13}{18} = 1.625$

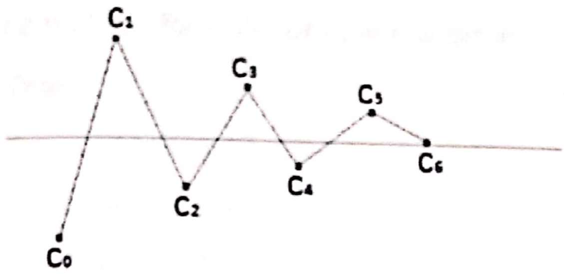
C_6 = [1; 1, 1, 1, 1, 1, 1] $=\frac{21}{13} = 1.6153846154$

Some interesting observations:

⇒These convergent ck approach the actual value 21/13 as k increases, where $0 \le k \le 6$.

⇒The convergent c_{2k} approach it from below and the convergent C_{2k+1} from above, so the convergent are alternately less than and greater then 21/13 except the last convergence that is,

$c0 < c2 < c4 < 21/13 < c5 < c3 < c1$;



⇒The convergent display a remarkable pattern

⇒$ck = \frac{F_{k+2}}{F_{k+1}}$, $0 \le k \le 6$

THEOREM: The k^{th} convergents of the finite simple continued fraction $[a_0, a_1, a_2, \dots, a_k]$ is

$ck = \frac{p_k}{q_k}$ where $2 \le k \le n$ and the sequence $\{p_k\}$ and $\{q_k\}$ are defined recursively as follows:

$p_0 = a_0 \qquad q_0 = 1$

$p_1 = a_0a_1 + 1 \qquad q_1 = a_1$

$p_k = a_kp_{k-1} + p_{k-2} \qquad q_k = a_kq_{k-1} + q_{k-1}$

Proof: We shall prove by induction that $ck = p_k/q_k$ yields the k^{th} convergents of the continued fraction for each value of k, where $0 \le k \le n$.

When $k = 0$,

$c_0 = [a_0] = \frac{a_0}{1} \qquad = \frac{p_0}{q_0}$

When k=1

$C_1 = [a_0; a_1] = a_0 + \frac{1}{a_0} = \frac{a_0a_1+1}{a_1} = \frac{p_1}{q_1}$

Thus, the theorem is true when $k = 0$ and $k = 1$.

Now assume that the formula for c_k works for an arbitrary integer m, where $2 \leq m \leq n$. That is,

$$C_m = \frac{p_m}{q_m} = \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + q_{m-2}} \quad \text{.........(1)}$$

Then,

$$C_{m+1} = [a_0, a_1, a_2, \ldots\ldots a_m, a_{m+1}]$$

$$= [a_0, a_1, a_2, a_3, \ldots\ldots a_{m-1}, a_m + \frac{1}{a_{m+1}}]$$

Notice that the integers p_{m-1}, p_{m-2}, q_{m-1}, q_{m-2} depend only on the partial quotients a_0, a_1, a_2, a_3, … , a_{m-1} and not on a_m so the convergents c_{m+1} can be computed from equation (1) by replacing a_m with $a_m + \frac{1}{a_{m+1}}$:

$$C_{m+1} = \frac{\left(a_m + \frac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \frac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}}$$

$$= \frac{a_{m+1}\,(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}}$$

$$\frac{a_{m+1}\,(p_m) + p_{m+1}}{a_{m+1}\,(q_m) + q_{m-1}}$$

Thus, by induction, the equation (1) works for every value of k, where $0 \leq k \leq n$.

EXAMPLE 1: Using the above theorem, compute the convergents of the continued fraction [2; 3, 1, 5] =52/23

Solution: We have $a_0 = 2$, $a_1 = 3$, $a_2 = 1$, $a_3 = 5$, First, we compute p_k and q_k for each k, where $0 \leq k \leq 3$:

$p_0 = a_0 = 2 \qquad q_0 = 1$

$p_1 = a_0 a_1 + 1 = 2 \times 3 + 1 = 7$ $\quad q_1 = a_1 = 3$

$p_2 = a_2 p_1 + p_0 = 1 \times 7 + 2 = 9$ $\quad q_2 = a_2 q_1 + q_0 = 1 \times 3 + 1 = 4$

$p3 = a_3 p_2 + p_1 = 5 \times 9 + 7 = 52$ $\quad q_3 = a_3 q_2 + q_1 = 5 \times 4 + 3 = 23$

Thus, the various convergents are

$$C_0 = \frac{p_0}{q_0} = \frac{2}{1} \qquad C_1 = \frac{p_1}{q_1} = \frac{7}{3}$$

$$C_2 = \frac{p_2}{q_2} = \frac{9}{4} \qquad C_3 = \frac{p_3}{q_3} = \frac{52}{23}$$

THEOREM: Let $c_k = p_k/q_k$ be the k^{th} convergent of the simple continued fraction $[a_0, a_1, a_2, \ldots]$, where $1 \leq k \leq n$. Then

$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$

Proof: Using the definition of the sequence $\{p_k\}$ and $\{q_k\}$ in the preceding theorem,

$p_1 q_0 - q_1 p_0 = (a_0 a_1 + 1) \times 1 = (-1)^{1-1}$

So, the formula works when k=1.

Now assume that it is true for an arbitrary positive integer $k < n$

$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$

Then, by the recursive definition of p_k and q_k,

$p_{k+1} q_k - q_{k+1} p_k = (a_{k+1} p_k + p_{k+1}) q_k - (a_{k+1} q_k + q_{k-1}) p_k$

$= -(p_k q_{k-1} - q_k p_{k-1})$

$= -(-1)^{k-1}$, by the inductive hypothesis

$= (-1)^k$

So the formula works for $k + 1$ also. Thus, by induction, the theorem is true for every positive integer $\leq n$.

EXAMPLE 2: Verify the above theorem, using the convergent of the continued fraction [2; 3,1,5]

Solution: Using example (1), we have

$p_1q_0 - q_1p_0 = 7 \times 1 - 3 \times 2 = 1 = (-1)^{1-1}$

$p_2q_1 - q_2p_1 = 9 \times 3 - 4 \times 7 = -1 = (-1)^{2-1}$

$p_3q_2 - q_3p_1 = 52 \times 4 - 23 \times 9 = 1 = (-1)^{2-1}$

Thus, $p_k\, q_{k-1} - q_k\, p_{k-1} = (-1)^{k-1}$ for every value of k, where1 $\leq k \leq 3$.

INFINITE CONTINUED FRACTIONS

DEFINITION: Suppose there are infinitely many terms in the expression $[a0; a1, \ldots, , \ldots]$ where $a_0 \geq 0$ and $a_i > 0$ for $i \geq 1$. Such a fraction is an infinite continued fraction. In particular, if each a_j is an integer, then it is an integer, then it is infinite simple continued fraction.

More generally, an infinite simple continued fraction is of the form

$$X = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ldots}}}$$

where $a_0 \geq 0$, and a_i and b_{i+1} are integers for each *i*.

An interesting example of such a continued fraction is the identity for $\frac{4}{\pi}$ discovered in (1655) by Lord William V. Brouncker (1620-1684), the first president of Royal society. He discovered it by converting Walli"s celebrated infinite product

$$\frac{4}{\pi} = \frac{3\times3\times5\times5\times7\times7\ldots.}{2\times4\times4\times6\times6\ldots}$$

Into a continued fraction

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

This is the first recorded infinite continued fraction, but Brouncker did not provide a proof,it was given by Euler in 1775.

An infinite continued fraction for $\frac{\pi}{4}$ is

$$\frac{\pi}{4} = 1 + \cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\cdots}}}}$$

In 1999, L.J. Lange of the University of Missouri developed an equally fascinating continued fraction for π

$$\pi = 3 + \cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\cdots}}}}$$

For convenience, we restrict our discussion to infinite simple continued fraction, where, $b1$ =1. The simplest such continued fraction is [1; 1, 1, 1, ...] One of the most astounding continued fraction was developed by the Indian mathematical genius Srinivasa Ramanujan, who studied them in 1908

$$(\sqrt{\sqrt{5\alpha}} - \alpha)e^{\frac{2\pi}{5}} = [0; e^{-2\pi}, e^{-4\pi}, e^{-6\pi}, \ldots..]$$

Where 'α' denotes the golden ratio. When Ramanujan communicated this marvelous result to Hardy in his letter in 1913, Hardy was stunned by the discovery and could not derive it himself.

Equally intriguing is its reciprocal:

$$\frac{e^{\frac{-2\pi}{5}}}{\sqrt{\sqrt{5\alpha}-\alpha}} = [1; e^{-2\pi}, e^{-4\pi}, e^{-6\pi}, \ldots\ldots]$$

Ramanujan discovered about 200 such infinite continued fractions.

EXAMPLE 1: Express 3 is an infinite simple continued fraction.

Solution:

Let $x = x_0 = 3$ and $a_0 = \sqrt{3} \cong 1$ then,

$$X_1 = \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{\sqrt{3}+1}{2} \rightarrow a_1 = [x_1] = 1$$

$$X_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{\sqrt{3}+1}{2}-1} = \frac{1}{\frac{\sqrt{3}-1}{2}} = \frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{2(\sqrt{3}+1)}{2} = \sqrt{3}+1 \cong 2$$

$\rightarrow a_2 = [x_2] = 2$

$$X_3 = \frac{1}{\sqrt{3}+1-2} = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{\sqrt{3}+1}{2} \rightarrow a_3 = [x_3] = 1$$

$$X_4 = \frac{1}{x_3 - a_3} = \frac{1}{\frac{\sqrt{3}-1}{2}-1} = \frac{1}{\frac{\sqrt{3}-1}{2}} = \frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{2(\sqrt{3}+1)}{2}$$

$= \sqrt{3}+1 \cong 2 \rightarrow a_4 = [x_4] = 2$

Clearly the pattern continues, then

$\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \ldots]$

Then often written as

$3 = [1; 1,2]$

EXAMPLE 2: Express $\sqrt{7}$ as an infinite simple continued fraction.

Solution:

Let $x = x_0 = \sqrt{7}$ and $a_0 = \sqrt{7} \cong 2$ then,

$$X_1 = \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{(\sqrt{7}-2)(\sqrt{7}+2)} = \frac{\sqrt{7}+2}{3} \rightarrow a_1 = [x_1] = 1$$

$$X_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{\sqrt{7}+2}{3}-1} = \frac{1}{\sqrt{7}-1} = \frac{3(\sqrt{7}+1)}{6} = \frac{\sqrt{7}+1}{2} \rightarrow a_2 = [x_2] = 1$$

$$X_3 = \frac{1}{x_2 - a_2} = \frac{1}{\frac{\sqrt{7}+1}{2}-1} = \frac{1}{\sqrt{7}-1} = \frac{3(\sqrt{7}+1)}{6} = \frac{\sqrt{7}+1}{3} \rightarrow a_3 = [x_3] = 1$$

$$X_4 = \frac{1}{x_3 - a_3} = \frac{1}{\frac{\sqrt{7}+1}{3}-1} = \frac{1}{\sqrt{7}-2} = \frac{3(\sqrt{7}+1)}{3} = \sqrt{7}+2 \rightarrow a_4 = [x_4] = 4$$

Thus, pattern continues,

Clearly $\sqrt{7}$ = [2; 1, 1, 1, 4, 1, 1, 1, 4, ...] we write it as

$\sqrt{7} = [2; \overline{1, 1, 1, 4}]$

Let us consider the continued fractions $[a_0; a_1, a_2, \dots, a_n]$ is infinite then the convergents $c_n = [a_0; a_1, a_2, \dots, a_n]$ are finite and hence represents rational numbers so the properties of convergent can be applied to these convergent since

$c_0 < c_2 < c_4 < \cdots < c_5 < c_3 < c_1$

The sequence $\{c_{2n}\}$ is an increasing sequence that is bounded above by c_1, and the sequence $\{c_{2n+1}\}$ is a decreasing sequence that is bounded below by c_0 consequently both sequence have limits ; that is as 'n' approaches infinity ; sequence $\{c_{2n}\}$ approaches a limit l; the sequence $\{c_{2n+1}\}$ approaches a limit l^1;

Thus,

$lim_{n\to\infty} c_{2n} = l$ and $lim_{n\to\infty} c_{2n+1} = l^1$

THEOREM : Let $c_k = [a_0; a_1, a_2, a_3, \dots, a_k]$ denote the k^{th} convergent of the simple continued fraction $[a_0; a_1, a_2, a_3, \dots]$. Then $lim_{n\to\infty} c_{2n} = lim_{n\to\infty} c_{2n+1}$

Proof: On the basis of above relation

$$p_{2n+1}q_{2n} - q_{2n+1}p_{2n} = (-1)^{2n}$$

We have

$$C_{2n+1} - C_{2n} = \frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}}$$

$= \frac{p_{2n+1}q_{2n} - q_{2n+1}p_{2n}}{q_{2n+1}q_{2n}}$

$= \frac{(-1)^{2n}}{q_{2n+1}q_{2n}}$

$< \frac{1}{q_{2n}^2}$, since $q_{2n+1} > q_{2n}$

As n gets larger and larger q_n, and hence q_n^2 gets larger and larger ; then $\frac{1}{q_{2n}^2}$ gets smaller and smaller, but never negative. So $\lim_{n\to\infty}(c_{2n+1} - c_{2n}) = 0$.

Thus,

$= \lim_{n\to\infty} c_{2n+1} - \lim_{n\to\infty} c_{2n}$

$= \lim_{n\to\infty}(c_{2n+1} - c_{2n})$

$= 0$

So the two limits are equal.

THEOREM : The value of any infinite continued fraction is an irrational number. Or the infinite simple continued fraction $[a0; a1, a2, \ldots]$ represent an irrational number.

Proof: Suppose that x denotes the values of the infinite continued fraction $[a0, a1, a2, \ldots]$; that is x is the limit of the sequence of convergents.

$$cn = [a_0, a_1, a_2, \ldots a_n] = \frac{p_n}{q_n}$$

Because 'x' lies strictly between the successive convergents cn and $cn+1$,

We have

$$0 < | x - c_n| < | c_{n+1} - c_n | = |\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}| = \frac{1}{q_n q_{n+1}}$$

With the view to obtaining a contradiction assume that x is a rational number say $x = a/b$ where a and $b > 0$ are integers then

$$0 < |\frac{a}{b} - \frac{p_n}{q_n}| = \frac{1}{q_n q_{n+1}}$$

And so, upon multiplication by the positive number b qn

$$0 < |aqn - bpn| < \frac{b}{q_{n+1}}$$

We call that the values of qi , increase without bound as i increase if n is chosen so large that $b <$ qn+1, the result is 0 < $|aqn - bpn|$ < 1 this says that there is a positive integer namely $|aqn - bpn|$ between 0 and 1 an obvious impossibility.

Hence x cannot be a rational number in other words, x is an irrational number
This completes the proof.

THEOREM: If the infinite continued fraction $[a_0; a_1, a_2, ...]$ and $[b_0; b_1, b_2, ...]$ are equal then

$$a_n = b_n \text{ for all } n \geq 0.$$

Proof: If $x = [a_0; a_1, a_2, ...]$ then $c_0 < x < c_1$ which is the same as saying that $a_0 < x < a_0 + \frac{1}{a_1}$ knowing that the integer $a_1 \geq 1$ this produces the inequality $a_0 < x < a_0 + 1$. Hence, $[x] = a_0$, where [x] , is the traditional notation for the greatest integer or "bracket" function.

Now assume that $[a_0; a_1, a_2, ...] = x = [b_0; b_1, b_2, ...]$ or to put it in a different form

$$a_0 + \frac{1}{[a_1; a_2, ...]} = x = b_0 + \frac{1}{[b_1; b_2, ...]}$$

By virtue of the conclusion of the first paragraph, we have $a_0 = x = b_0$, from which it may then be deduced that $[a_1; a_2, ...] = [b_1; b_2, ...]$ when the reasoning is repeated, we next conclude that $a_1 = b_1$ and that $[a_2; a_3, ...] = [b_2; b_3, ...]$ the process continues by mathematical induction.

There by giving $an = bn$ for all $n \geq 0$

This completes the proof.

EXAMPLE 1: Find the infinite continued fraction expansion for π

Solution: Let us consider the value of π = 3.142592653 defined by the ancient Greeks as the ratio of the circumference of a circle to its diameter the letter π, from the Greek word perimeteros, was never employed in antiquity for this ratio it was Euler"s adoption of the symbol in his many popular text books that made it widely known and used.

Let $x_0 = \pi$, $a_0 = \pi \cong 3$ $a_0 = 3$

$$X_1 = \frac{1}{x_0 - a_0} = \frac{1}{\pi - 3} = \frac{1}{0.142592653} = 7.06251330 \qquad a_1 = 7$$

$$X_2 = \frac{1}{x_1 - a_1} = \frac{1}{7.06251330 - 7} = \frac{1}{0.6251330} = 15.99659440 \qquad a_2 = 15$$

$$X_3 = \frac{1}{x_2 - a_2} = \frac{1}{15.99659440 - 7} = \frac{1}{0.99659440} = 1.0031723 \qquad a_3 = 1$$

$$X_4 = \frac{1}{x_3 - a_3} = \frac{1}{1.00341723 - 7} = \frac{1}{0.00341723} = 292.63467 \qquad a_4 = 292$$

And continues this pattern, we get

π = [3;7,15,1,292,.........]

PELL'S EQUATION

What little action Fermat took to publicize his discoveries came in the form of challenges to other mathematicians. Perhaps he hoped in this way to convince them that his new style of number theory was worth pursuing. In January of 1657, Fermat proposed as a challenge to the European mathematical community – thinking probably in the first place of John Wallis, England's most renowned practitioner before Newton – a pair of problems:

1. Find a cube which increased by the sum its proper divisors, becomes a square; for example,

$$7^3 + 1 + 7 + 7^2 = 20^2.$$

2. Find a square which, when increased by the sum of proper divisors, becomes a cube.

On hearing of the contest, Fermat's favorite correspondent, Bernhard Frenicle de Bessy, quickly supplied a number of answer to the first problem; typical of these is $(2 \,.\, 3 \,.\, 5 \,.\, 13 \,.\, 41.47)^3$, which when increased by the sum of its proper divisors becomes $(2^7 \,.\, 3^2 \,.\, 5^2 \,.\, 7 \,.\, 13 \,.\, 1.29)^2$. While Frenicle advanced to solutions in still larger composite numbers, Wallis dismissed the problems as not worth his effort, writing, "Whatever the details of the matter, it finds me too absorbed by numerous occupations for me to be able to devote my attention to it immediately; but I can make at this moment this response: The number 1 in and of itself satisfies both demands. Barely concealing his disappointment, Frenicle expressed astonishment that a mathematician as experienced as Wallis would have made only the trivial response when, in view of Fermat"s stature, he should have sensed the problem"s greater depths.

Fermat"s interest, indeed, lay in general methods, not in the wearying computation of isolated cases. Both Frenicle and Wallis overlooked the theoretic aspect that the challenge problems were meant to reveal on careful analysis. Although the phrasing was not entirely precise, it seems clear that Fermat had intended the first of his queries to be solved for cubes of prime numbers. To put it otherwise, the problem called for finding all integral solutions of the equation

$$1 + x + x^2 + x^3 = y^2$$

Or equivalently,

$$(1 + x)(1 + x^2) = y^2$$

where x is an odd integer. Because 2 is the only prime that divides both factors on the left-hand side of this equation, it may be written as

But if product of two relatively prime integers is a perfect square, then each of them must be a square; hence, $a = u2$, $b = v2$ for some u and v, so that

$1+x = 2a = 2u^2 \quad 1+x^2 = 2b = 2v^2$

This means that any integer x that satisfies Fermat"s first problem must be a solution of the pair of equations

$x = 2u^2 - 1 \quad x^2 = 2v^2 - 1$

the second being a particular case of equation $x^2 = dy^2 \pm 1$

In February 1657, Fermat issued his second challenge, dealing directly with the theoretic point at issue: Find a number y that will make $dy2 + 1$ a perfect square, where d is a positive integer that is not a square; for example, 3. 12 + 1 = 22 and 5. 42 + 1 = 92. If, said Fermat, a general rule cannot be obtained, find the smallest values of y that will satisfy the equations. $61y^2+1= x^2$; or $101y^2 + 1 = -x^2$. Frenicle proceeded to calculate the smallest positive solution of $x^2 - dy^2 = 1$ for all permissible values of d to 150 and suggested that Wallis extend the table to $d = 200$ or at least solve $x^2 - 151y^2 =1$ and $x^2 - 313y^2 =1$, hinting that the second equation might be beyond Wallis"s ability. In reply, Willis"s patron Lord William Brouncker of Ireland stated that it had only taken him an hour or so to discover that

$$(126862368)^2 - 313(7170685)^2 = -1$$

and therefore $y = 2.7170685 . 126862368$ gives the desired solution to $x^2-313y^2 = 1$; Wallis solved the other concrete case, furnishing

$$(1728148040)^2 - 151(140634693)^2 = 1$$

The size of these number in comparison with those arising from other values of d suggests that Fermat was in possession of a complete solution to the problem, but this was never disclosed (later, he affirmed that his method of infinite descent had been used with success to show the existence of an infinitude of solutions of $x^2-dy^2=1$). Brouncker, under the mistaken impression that rational and not necessarily integral values were allowed, had no difficulty in supplying an answer; he simply divided the relation

$$(r^2 + d)^2 - d\,(2r)^2 = (r2 - d)2$$

By the quantity $(r2 - d)2$ to arrive at the solution

$$x = \frac{r^2+d}{r^2+d} \qquad y = \frac{2r}{r^2-d}$$

where $r \neq d$ is an arbitrary rational number. This, needless to say was rejected by Fermat, who wrote that "solution in fractions, which can be given at once from the merest elements of arithmetic, do not satisfy me." Now informed of all the conditions of the challenge, Brouncker and Wallis jointly devised a tentative method for solving $x^2 - dy^2 = 1$ in integers, without being able to give a proof that it will always work apparently the honors rested without Brouncker, for Wallis congratulated Brouncker with some pride that he had "preserved untarnished the frame that Englishmen have won in former times with Frenchmen."

Theorem 1: If , q is positive solution $x^2 - dy^2 = 1$ of then p/q is convergent of the continued fraction expansion of $\sqrt{d}$.

Proof: In light of the hypothesis that $p^2 - dy^2 = 1$

we have $(p - q\sqrt{d})(p + q\sqrt{d}) = 1$ implying that $p > q$ as well as that

$$\frac{p}{q} - \sqrt{d} = \frac{1}{q(p+q(\sqrt{d}))} \text{ as a result.}$$

$$0 < \frac{p}{q} - \sqrt{d} < \frac{\sqrt{d}}{q(q\sqrt{d}+q\sqrt{d})} = \frac{\sqrt{d}}{2q^2\sqrt{d}} = \frac{1}{2q^2}$$

$$= \left|\frac{p}{q} - \sqrt{d}\right| < \frac{1}{2q^2}.$$

Theorem: If p/q is convergent of the continued fraction expansion of $\sqrt{d}$,

Then $x =$, $y = q$, is a solution of one of the equations $x^2 - dy^2 = 1$

where $|k| < 1 + 2\sqrt{d}$.

Proof: If p/q is convergent of $\sqrt{d}$, by previous theorem,

We have $\left|\frac{p}{q} - \sqrt{d}\right| < \frac{1}{2}$

and therefore $|p - q\sqrt{d}| < \frac{1}{q^2}$...........(1)

This being so,we have $|p + q\sqrt{d}| = |(p - q\sqrt{d}) + (2q\sqrt{d})|$

$\leq |p - q\sqrt{d}| + |2q\sqrt{d}|$

$< \frac{1}{q} + 2q\sqrt{d}$

$\leq (1 + 2\sqrt{d})q$ ....(2)

These two inequality combine to yield

$|p^2 - dq^2| = |(p - q\sqrt{d})|\ |(p + q\sqrt{d})|$

$< \frac{1}{q}(1 + 2\sqrt{d})q$

$= 1 + 2\sqrt{d}$

This completes proof.

EXAMPLE: Find the positive solution of following equation

$x^2 - 13y^2 = -1$

Solution: Let us take $d = 13$, using the continued fraction expansion of

13 = [3; 1 , 1 , 1 , 1 , 6] that is

$$\sqrt{13} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cdots}}}}}$$

The first few convergent of 13 are determined to be $\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \ldots\ldots$

running through the calculation of $p_n^2 - 13q_n^2$ we find that

$9 - 13(1) = -4$

$16 - 13(1) = 3$

$49 - 13(4) = -3$

$121 - 13(9) = 4$

$324 - 13(25) = -1$

$\therefore x = 18$, $y = 5$ provides a positive solution of the equation $x^2 - 13y^2 = -1$

APPLICATIONS

1) Square roots

Generalized continued fractions are used in a method for computing squre roots. The identiy,

$\sqrt{x} = 1 + \frac{X-1}{1+\sqrt{X}}$ -------------1

leads via recursion to the generalized continued fraction for any square root.

$\sqrt{x} = 1 + \cfrac{X-1}{2+\cfrac{X-1}{2+\cfrac{X-1}{2+\cdots}}}$ ---------2

2) Pells equation

Continued fraction play an essential role in the solution of pells equation. For example, for positive integers p and q and non-square n, it is true that $p^2 - nq^2 = \pm 1$ if and only if p/q is a convergent of the regular continued fraction for $\sqrt{n}$.

3) Eigen values eigenvectors

The lancoz algorithm uses a continued fraction expansion to iteratively approximate the eigenvalues and eigenvectors of a large sparse matrix.

4) Dynamical system

Continued fraction also play a role in the study of dynamical systems,

Where they tie together the farey fractions which are seen in the Mandelbrot set with minkowskis question mark function and the modular group gamma.

The backward shift operator for continued fraction is the map $h(x) = \frac{1}{x} - \lfloor 1/x \rfloor$ called the Gauss map, which lops off digits of a continued fraction expansion: $h[0;a_1,a_2,a_3,\ldots\ldots] = [0;a_2,a_3,\ldots\ldots]$ The transfer operator of this map is called the gauss kuzmin wirsing operator. The distribution of the digits in continued fractions is given by the zeroth eigenvector of this operator, and is called the Gauss-kuzamin distribution.

CHAPTER 2

EXPANSION OF IRRATIONAL NUMBER

INTRODUCTION

So far our discussion has been limited to the expansion of rational numbers. We proved that a rational number can be expanded into a finite simple continued fraction, and, conversely, every finite simple continued fraction represents a rational number. This chapter will deal with the simple continued fraction expan- sion of irrational numbers, and we shall see that these fractions do not terminate but go on forever. An irrational number is one which cannot be represented as the ratio of two integers. The numbers

$$\sqrt{2}\ ,\ \sqrt{3}\ ,1\pm\sqrt{2}\ ,\ \frac{1\pm\sqrt{3}}{5}$$

are all irrational. Any number of the form

$$\frac{P\pm\sqrt{D}}{Q}$$

where P, D, Q are integers, and where D is a positive integer not a perfect square, is irrational. A number of this form is called a quadratic irrational or quadratic surd since it is the root of the quadratic equation

$$Q^2x^2\text{-}2PQx\text{+}(P^2\text{-}D)\text{=}0.$$

Our discussion will be limited to the expansion of quadratic irrationals.

There are irrational numbers which are not quadratic surds. The irrational number ? π= 3.14159 ... is one example. The irrational number $\sqrt{2}$ is the solution of the algebraic equation $x^2 - 2 = 0$, and is therefore called an "algebraic number". An algebraic number is a number x which satisfies an algebraic equation, i.e., an equation of the form

$$a_0x^n{}_{+}a_1x^{n\text{-}1}\text{+},\ldots\text{+}a_n\text{=}0,$$

where a_o, a_l, ... are integers, not all zero. A number which is not algebraic is called a transcendental number. It can be proved that π is transcendental, but this not easy to do.1 The number e is also transcendental. It is quite difficult to expand transcendental numbers into continued fractions; using decimal approximations to these ... ..., numbers, such as π = 3.14159 and e = 2.71828 we can calculate a few of the first terms of their continued fraction expansions, but the methods of obtaining the expansions of π and e given in Appendix I1 are beyond the scope of this monograph. Those who wish to learn about the two classes of irrational numbers, namely algebraic irrational numbers and transcendental numbers, and to study the deeper properties of each should read the first monograph in the NML (New Mathematical Library) series: Numbers: Rational and Irrational, by Ivan Niven.

PRELIMINARY EXAMPLES

The procedure for expanding an irrational number is fundamentally the same as that used for rational numbers. Let x be the given irrational number. Calculate al, the greatest integer less than x, and express x in the form

$$x = a_1 + \frac{1}{x_2}, \qquad 0 < \frac{1}{x_2} < 1,$$

Where the number

$$x_2 = \frac{1}{x - a_1} > 1$$

is irrational; for, if an integer is subtracted from an irrational number, the result and the reciprocal of the result are irrational. To continue, calculate a_2, the largest integer less than x_2, and express 22 in the form

$$x_2 = a_2 + \frac{1}{x_3}, \quad 0 < \frac{1}{x_3} < 1, a_2 \geq 1,$$

where, again, the number

$$x_3 = \frac{1}{x_2 - a_2} > 1$$

is irrational.

This calculation may be repeated indefinitely, producing in succession the equations

$$x = a_1 + \frac{1}{x_2}, \qquad x_2 > 1,$$
$$x_2 = a_2 + \frac{1}{x_3}, \qquad x_3 > 1, \quad a_2 \geq 1$$
$$x_3 = a_3 + \frac{1}{x_4}, \qquad x_4 > 1, \quad a_3 \geq 1,$$
$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots \qquad \text{.......................(1)}$$
$$x_n = a_n + \frac{1}{x_{n+1}}, \quad x_{n+1} > 1, \quad a_n \geq 1,$$
$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots$$

where al, as, a,, are all integers and where the numbers b ... x, x_2, x_3, x_4, are all irrational. This process cannot terminate, for ' the only way this could happen would be for some integer an to be equal to x,,, which is impossible since each successive x, is irrational. Substituting xz from the second equation in (1) into the first equation, then x3 from the third into this result, and so on, produces the required infinite simple continued fraction

$$x = a_1 + \frac{1}{x_2} = a_1 + \frac{1}{a_2 + \frac{1}{x_3}} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{x_4}}} = \cdots,$$

Or

$$x = [a_1, a_2, a_3, a_4 \dots],$$

where the three dots indicate that the process is continued indefinitely.

Before discussing some of the more "theoretical" aspects of infinite simple continued fractions, an example or two should be worked to make sure the expansion procedure is understood.

EXAMPLE 1: Expand $\sqrt{2}$ into an infinite simple continued fraction.

SOLUTION: The largest integer < $\sqrt{2}$ = 1.414 . . . is $a_1 = 1$, so

$$\sqrt{2} = a_1 + \frac{1}{x_2} = 1 + \frac{1}{x_2}.$$

Solving this equation for x_2, we get

$$x_2 = \frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2} + 1.$$

Consequently,

$$\sqrt{2} = a_1 + \frac{1}{x_2} = 1 + \frac{1}{\sqrt{2}+1}.$$

The largest integer < $x_2 = \sqrt{2} + 1$ = 2.414 . . . is $a_2 = 2$, s

$$x_2 = a_2 + \frac{1}{x_3} = 2 + \frac{1}{x_3},$$

Where

$$x_3 = \frac{1}{x_2 - 2} = \frac{1}{(\sqrt{2}+1)-2} = \frac{1}{\sqrt{2}-1}$$

$$= \frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2} + 1 > 1.$$

At this stage we know that

$$\sqrt{2} = a_1 + \frac{1}{x_2} = 1 + \frac{1}{2+}$$

Since $x_3 = \sqrt{2} + 1$ is the same as $x = \sqrt{2} + 1$, the calculations of $x_4, x_5, \dots$ will a11 produce the same result, namely $\sqrt{2} + 1$. Thus all the subsequent partial quotients will be equal to 2 and the infinite expansion of $\sqrt{2}$ will be

$$\sqrt{2} = 1 + \frac{1}{2+} \frac{1}{2+\cdots} = [1,2,2,2, \dots] = [1,\overline{2}].$$

The bar over the 2 on the right indicates that the number 2 is repeated over and over.

Immediately some questions are raised. For example, is it possible to prove that the infinite continued fraction [l, 2, 2, . . .] = $[1,\bar{2}]$,actually represents the irrational number $\sqrt{2}$? Certainly there is more to this than is evident at first glance, and it will be one of the more difficult questions to be discussed in this chapter. We can, however, give a formal answer to this question. A formal answer means, roughly speaking, that we go through certain manipulations, but no claim is made that every move is necessarily justified. With this understanding, we write

$$X = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

Or

$$X - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

Hence

$$X = 1 + (x - 1),$$

or 1 = 1, which tells us nothing about x. However, using the same idea, we can write

$$x = 1 + \cfrac{1}{2 + \left(\cfrac{1}{2 + \cfrac{1}{2 + \ddots}}\right)}$$

$$= 1 + \frac{1}{2 + (x-1)} = 1 + \frac{1}{x+1}$$

from which we see that

$$x - 1 = \frac{1}{x-1},$$

So

$$(x - 1)(x + 1) = 1, \text{ or } x^2 = 2,$$

Thus

$$X = 1 + \frac{1}{2} + \frac{1}{2} + \dots = \sqrt{2}$$

Some additional examples of a similar sort are:

$$\sqrt{3} = [1,1,2,1,2,1,2, \dots] = [1,1,2],$$

$$\sqrt{15} = [3,1,6,1,6, \dots] = [3,1,6],$$

$$\sqrt{31} = [51,1,3,5,3,1,1,10].$$

In each of these examples the numbers under the bar form the periodic part of the expansion, the number 1/31 having quite a long period.

Irrational number:-

An irrational number is a non terminating non repeating decimal.

For example: A example of irrational number are $\pi, \sqrt{2}, \sqrt{3}$ *and* $\sqrt{5}$*etc* .....

The value of $\pi, \sqrt{2}$ *and* $\sqrt{3}$ are shown below to 50 decimal places.

$\pi \approx 3.14159265358979328462 64$..........

$\sqrt{2} \approx 1.41421356237309504880 1688$......

$\sqrt{3} \approx 1.73205080756887729352 74463$....

Note: If we solve an equation such as $x^2=25$ we take the square root of both sides and obtained a solution of $x= \pm 5$.HOwer the "$\sqrt{\ }$ " symbol denote the principle square root and represents only the positive square root.

Therefore we say for example : $\sqrt{25} = 5$, Not $\sqrt{25} = \pm 5$.

➢ "The sum of two irrational number is irrational number"

Example:$2\sqrt{7}, \sqrt{3}$.

➢ "The product of two irrational number is irrational number"

Example: $\sqrt{2} \times \sqrt{8}$ is rational but $\sqrt{3} \times \sqrt{8}$ is irrational.

➢ "The circumference of a circle is irrational"

➢ "The diagonal of a square is irrational"

Example: Prove that $\sqrt{3}$ is an irrational number ?

Proof: Let us assume $\sqrt{3}$ is a rational number where a and b are two positive integer which can be represent in the form is

$\sqrt{3}$=a/b (because a and b are co-prime)

Squaring on both sides

$(\sqrt{3})^2 = (a/b)^2$

$3 = a^2/b^2$

$3b^2 = a^2$

$b^2 = a^2/3$..........................(1)

∴ 3 divides a^2

3 divides a

Assume, a = 3c where c is an integer

Squaring on both sides ,we get

$a^2 = (3c)^2$

$a^2 = 9c^2$

$3b^2 = 9c^2$

$\frac{3b^2}{9} = c^2$

$\frac{b^2}{3} = c^2$

∴ 3 divides b^2

3 divides b

From above conclusion a & b are common number 3 which is contradicts with that a & b are co-prime our assumption is wrong.

∴ $\sqrt{3}$ is an irrational number.

Example: Prove that $\sqrt{2}$ is an irrational number?

Proof: Let us assume $\sqrt{2}$ is a rational number where a and b are two positive integer which can be represent in the form is

$\sqrt{2}$=a/b (because a and b are co-prime)

Squaring on both sides

$(\sqrt{2})^2 = (a/b)^2$

$2 = a^2/b^2$

$2b^2 = a^2$

$b^2 = a^2/2$.......................(1)

∴ 2 divides a^2

2 divides a

Assume, a = 2c where c is an integer

Squaring on both sides ,we get

$a^2 = (2c)^2$

$a^2 = 4c^2$

$3b^2 = 4c^2$

$\frac{2b}{4} = c^2$

$\frac{2b2}{4} = c^2$

∴ 2 divide b^2

2 divide b

From above conclusion a & b are common number 3 which is contradicts with that a & b are co-prime our assumption is wrong.

∴ $\sqrt{2}$ is an irrational number.

Example: Prove that $\sqrt{5}$ is an irrational number ?

Proof: Let us assume $\sqrt{5}$ is a rational number where a and b are two positive integer which can be represent in the form is

$\sqrt{5}$=a/b (because a and b are co-prime)

Squaring on both sides

$(\sqrt{5})^2 = (a/b)^2$

$5 = a^2/b^2$

$5b^2 = a^2$

$b^2 = a^2/3$.......................(1)

∴ 5 divides a^2

5 divides a

Assume, a = 5c where c is an integer

Squaring on both sides ,we get

$a^2 = (5c)^2$

$a^2 = 25c^2$

$5b^2 = 9c^2$

$\frac{5b2}{25} = c^2$

$\frac{b2}{5} = c^2$

∴ 5 divide b^2

5 divide b

From above conclusion a & b are common number 3 which is contradicts with that a & b are co-prime our assumption is wrong.

∴ $\sqrt{5}$ is an irrational number.

Hence the proof.

Example: Show that $3\sqrt{2}$ is irrational number.

Proof: Let us assume, to the contrary that $3\sqrt{2}$ is rational number.

that is we find co-prime a and b ($\neq 0$).

Such that $3\sqrt{2} = a/b$

Since 3, a and b are integers a/3b is rational and so $\sqrt{2}$ is irrational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So we conclude that $3\sqrt{2}$ is irrational.

Example: Show that $5 - \sqrt{3}$ is irrational.

Proof: Let us assume, to the contrary that $5 - \sqrt{3}$ is irrational.

That is we can find co-prime a and b ($\neq 0$).

Such that $5 - \sqrt{3} = a/b$

$\therefore\ 5 - a/b = \sqrt{3}$

Rearranging this equation, we get

$\sqrt{3} = \frac{5b-a}{b}$

Since a and b are integer, we get $5 - a/b$ is rational and so $\sqrt{3}$ is irrational.

But this contradicts the fact that $\sqrt{3}$ is irrational. This contradicts has arises because of in correct assumption that $5 - \sqrt{3}$ is rational. So, we conclude that $5 - \sqrt{3}$ is irrational.

Diophantine equation

A great many puzzles, riddles, and trick questions lead to mathematical equations whose solutions must be integers. Here is a typical example: A farmer bought a number of cows at $80 each, and a number of pigs at $50 each. His bill was $810. How many cows and how many pigs did he buy?

If x is the number of cows and y the number of pigs, we have the equation

$$8x + 50y = 810 \quad \text{.................(1)}$$

Which is equivalent to

$$8x + 5y = 81 \quad \text{..................(2)}$$

If nothing limits the values of x and y in equation (2), we can give x any value, say x = $, and then solve the resulting equation

$$4 + 5y = 81$$

for y, getting $y = \frac{77}{5}$. In this sense, (2) is an indeterminate equation, which means that we can always find some value of y corre- sponding to any value we choose for x. If, however, we restrict the values of x and y to be integers, as the farmer is likely to do (since he is probably not interested in half a cow), then our example belongs to an extensive class of problems requiring that we search for integral solutions x and y of indeterminate equations. Indeterminate equations to be solved in integers (and sometimes in rational numbers) are often called Diophantine equations in honor of Diophantus, a Greek mathematician of about the third century A.D., who wrote a book about such equations. Our problem, it should be noted, has the further restriction that both x and y must not only be integers but must be positive. Equation (2) and hence equation (1) can be solved in many ways. In fact there is no harm in solving such equations by trial and error or by making intelligent guesses. For example, if we write equation (2) in the form

$$81 - 8x = 5y,$$

we need only search for positive integral values of x such that 81 – 8x is a multiple of 5. Letting x, in turn, take on the values 0, 1, 2, 3, . . . , 10, we find that x = 2 and x = 7 are the only non-negative values which make 81 - 82 a non-negative multiple of 5. The calculations are

$x = 2, \quad 81 - 8x = 81 - 16 = 65 = 5 \,.\, 13 = 5y, \quad y = 13,$

$x = 7, \quad 81 - 8x = 81 - 56 = 25 = 5 \,.\, 5 = 5y, \quad y = 5;$

hence the two solutions to our problem are (x, y) = (2, 13) and (x, y) = (7, 5). So the farmer could buy 2 cows and 13 pigs, or 7 cows and 5 pigs.

There are other ways of solving Diophantine equations. We shall give two additional methods. The first of these was used extensively by Euler in his popular text Algebra, published in 1770. The second method will show how the theory of continued fractions can be applied to solve such equations.

The Method Used Extensively by Euler

Let us consider again the equation

$$8x + 5y = 81 \quad \text{......................(3)}$$

Since y has the smaller coefficient, we solve the equation for y, getting

$$Y = \frac{81-8x}{5} \quad \text{......................(4)}$$

Both 81 and 8 contain multiples of 5, that is,

81 = 5.16+ 1 and 8=5.1+3;

therefore, from (2.4), we have

$$y = \frac{(5.16+1)-(5.1+3)\,x}{5} \text{.......(5)}$$

$$= (16 - x) + \frac{1 - 3x}{5}$$

$$= (16 - x) + t,$$

Where

$$t = \frac{1-3x}{5}$$

Or

$$3x + 5t = 1 \quad \text{............................(6)}$$

Since x and y must be integers, we conclude from equation (5) that t must be an integer. Our task, therefore, is to find integers x and t satisfying equation (6). This is the essential idea in Euler's method, i.e., to show that integral solutions of the given equation are in turn connected with integral solutions of similar equations with smaller coefficients. We now reduce this last equation to a simpler one exactly as we reduced (3) to (6). Solving (6) for x, the term with the smaller 'coefficient, we get

$$X = \frac{1-5t}{3} = \frac{1-(2.3-1)t}{3}$$

$$= -2t + \frac{t+1}{3}$$

$$= -2t + u \quad \text{...........(7)}$$

Where

$$u = \frac{t+1}{3},$$

Or

$$T = 3u - 1 \quad \text{...........................(8)}$$

Again, since x and t must be integers, u must also be an integer.

FAREY FRACTIONS

Another approach to approximating real numbers by rational uses what is known as Farey fractions, or the Farey sequence. For a positive integer n, these are defined as follows:

DEFINITION: The Farey fractions of order$n$, denoted F_n, are a set of rational numbers r/s with $0 \leq r \leq s \leq n$ and gcd(r,s) = 1. They are written in order of increasing size. The first few F_n are

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$

$$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$

$$F_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$$

Notice that the fractions occur in any n F will thereafter occur in any m F , for $m \geq n$.

Farey fractions have a curious history. The English geologist John Farey (1766 - 1826) published, without proof, several properties of this series of fractions in the Philosophical Magazine in 1816. The mathematician Augustin Cauchy saw the article and supplied the demonstrations later in same year, naming the fractions after Farey. It subsequently turned out.

that C.H.Haros had proved the results 14 years earlier, in the Journal de l"Ecole Polytechnique. Farey, of course, had never claimed to have proved anything.

We begin our investigation with one of the results stated by Farey but established earlier by Haros

THEOREM 1: If $\frac{a}{b} < \frac{c}{d}$ are consecutive fractions in Farey sequence F_n, then $bc - ad = 1$.

Proof: Because gcd (,b) = 1, the linear equation $bx - ay = 1$ has a solution $x = x_0$, $y = y_0$. Moreover, $x=x_0 + at$,$y = y_0 = bt$ will also be a solution for any integer **t**.

Choose $t=t_0$ so that

$$0 \le n - b < y_0 + bt_0 \le n$$

and set $x=x_0+bt_0, y=y_0+bt_0$. Since $y \le n$, x/y will be a fraction in F_n . Also

$$\frac{x}{y} = \frac{a}{b} + \frac{1}{by} > \frac{a}{b}$$

So that $\frac{x}{y}$ occurs later in the Farey sequence than $\frac{a}{b}$. If $\frac{x}{y} \neq \frac{c}{d}$, then $\frac{x}{y} > \frac{c}{d}$ and we obtain

$$\frac{x}{y} - \frac{c}{d} = \frac{dx - cy}{dy} \geq \frac{1}{dy}$$

as well as

$$\frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd} \geq \frac{1}{bd}$$

Adding the two inequalities gives

$$\frac{x}{y} - \frac{a}{b} \geq \frac{1}{dy} - = \frac{b+y}{bdy}$$

But $b + y > n$ (recall that $n - b < y$) and $d \le n$, resulting in the contradiction

$$\frac{1}{by} = \frac{bx - ay}{by} = \frac{x}{y} - \frac{a}{b} = \frac{b+y}{bdy} > \frac{n}{bdy} \geq \frac{1}{by}$$

Thus, $\frac{x}{y} = \frac{c}{d}$ and the equation $bx - ay = 1$ becomes $bc - ad = 1$.

If $\frac{a}{b} < \frac{c}{d}$ are two fractions in the Farey sequence F_n , we define their mediant fraction to be the expression $\frac{a+c}{b+d}$. Theorem allows us to concludee that the mediant lies between the given fractions. For the relations

$(b + d) - (a + c) = ad - bc < 0$

$(a + c)\, d - (b + d)\, c = ad - bc < 0$

Together imply that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Notice that if $\frac{a}{b} < \frac{c}{d}$ are consecutive fractions in n F ,and $b + d \le n$, then the median would be a member of F_n lying between them, an obvious contradiction. Thus, for successive fractions, $+ d \geq n + 1$.

It can be shown that those fractions that belongs to F_{n+1} but not to F_n are mediants of fractions in F_n . In passing from F_4 to F_5 , for instance, the new members are $\frac{1}{5}=\frac{0+1}{1+4}, \frac{2}{5}=\frac{1+1}{3+2}, \frac{3}{5}=\frac{1+2}{2+3}, \frac{4}{5}=\frac{3+1}{4+1}$

This enables one to build up the sequence F_{n+1} from F_n by inserting mediants with the appropriate denominator.

In using the mediant of two fractions in F_n to obtain a new member of F_{n+1} , the three fractions need not be consecutive in F_{n+1} (consider $\frac{1}{3}<\frac{3}{8}<\frac{2}{3}$ in F_8).we can say that if

$\frac{a}{b}<\frac{c}{d}<\frac{e}{f}$ are three consecutive fractions in any farey sequence ,then $\frac{c}{d}$ is the mediant of $\frac{a}{b}$ *and* $\frac{e}{f}$. For ,appealing once more to by Theorem the equations

$$bc-ad=1 \qquad de-cf=1$$

lead to $(a+e)d=c(b+f)$. It follows that $\frac{c}{d}=\frac{a+e}{b+f}$

Which is the median of $\frac{a}{b}$ *and* $\frac{e}{f}$ As an illustration, the three fractions $\frac{3}{8}<\frac{2}{5}<\frac{3}{7}$ are consecutive in F_8 with $\frac{2}{5}=\frac{3+3}{8+7}$.

Let us apply some of these ideas to show how an irrational number can be approximated, relatively well, by a rational number.

Theorem: For any irrational number $0 < x < 1$ and integer $n > 0$, there exists a fraction $\frac{u}{v}$ in such that $|x-\frac{u}{v}|<\frac{1}{v(n+1)}$.

Proof: In the farey sequence Fn,there are consecutive fraction $\frac{a}{b}<\frac{c}{d}$ such that either

$$\frac{a}{b}<x<\frac{a+c}{b+d} \text{ or } \frac{a+c}{b+d}<x<\frac{c}{d}$$

Where $\frac{a+c}{b+d}$ is the median of the two fractions. Because we know $bc-ad=1$ and $b+d \geq$ n+1, we can see that either

$$x-\frac{a}{b}<\frac{a+c}{b+d}-\frac{a}{b}=\frac{bc-ad}{b(b+d)}\leq\frac{1}{b(n+1)}$$

Or

$$\frac{c}{d}-x<\frac{c}{d}-\frac{a+c}{b+d}=\frac{bc-ad}{b(b+d)}\leq\frac{1}{d(n+1)}$$

Depending on the case, take $\frac{u}{v} = \frac{a}{b}$ or $\frac{u}{v} = \frac{c}{d}$.

This result can be extended beyond the unit interval.

Example 1: Let us determine a fraction $\frac{a}{b}$ with 0 <b< 5 such that $|\sqrt{7} - \frac{a}{b}| \leq \frac{1}{6b}$.

Solution: The greatest integer function yields [$\sqrt{7}$ – 2 = 0.64755... For the Farey sequenceF_5, the value 0.64755... lies in the interval between consecutive fractions $\frac{3}{5}$ and $\frac{2}{3}$. The median of the two fractions is $\frac{5}{8}$= 0.625 so that

$\frac{5}{8}$<0.64755... It follows from Theorem that

$|0.64755... - \frac{2}{3}| < \frac{1}{6.3}$

The argument employed in the corollary shifts this inequality into

$| 7 - \frac{8}{3}| < \frac{1}{6.3}$ So that $\frac{8}{3}$ is the fraction sought.

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A

PROJECT REPORT ON

"SOLUTION OF DIFFERENTIAL EQUATION"

HKES'S A V PATIL ARTS SCIENCE & COMMERCE COLLEGE ALAND"

Submitted to the

H.K.E. Society's

A V PATIL ARTS SCIENCE & COMMERCE COLLEGE ALAND"

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CERTIFICATE OF COMPLETION

This is to certify that the following students have successfully completed the project on **"SOLUTION OF DIFFERENTIAL EQUATION"** AT HKES'S A V PATIL ARTS SCIENCE & COMMERCE COLLEGE ALAND" is based on the project carried out under the guidance of laxman Rathod and is submitted to the Department of Science, H.K.E. Society's A V Patil Arts, Science & Commerce College Aland.

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INDEX

CHAPTER : - 01

SOLUTION OF DIFFERENTIAL EQUATION

1.1 INTRODUCATION

A differential equation is an equation that comprises both the function and its derivatives for any unknown function. Although the unknown function may be dependent on numerous derivatives in general, we will only look at the differential equation for a function of a single real variable in this course. The term **"ordinary differential equation"** refers to a type of equation like this.

A most general ODE has the Form

$F(x, y, y', \ldots, y^n) = 0$

Where

'F' is given function of n+2 variables & y=y(x) is an unknown function of a real variable x

Differential equations first came into existence with the inversion of calculus by Newton & Leibniz Isaac Newton listed three kinds of differential equations .

$\frac{dy}{dx}=f(x)$; $\frac{dy}{dx}=f(x,y)$; $x_1\frac{dy}{dx1} + x_2\frac{dy}{dx2}=y$

In all of these circumstances, y is an unknown function of x (or of x1 and x2), while f is a known function. He solves these and additional examples with infinite series, and discusses the non-uniqueness of the solutions.

Many disciplines of mathematics, as well as science and engineering, use ODEs. In most cases, a solution y(x) must be found explicitly or numerically to meet certain additional constraints. There are just a few types of ODEs for which all solutions can be found.

1.2 HISTROY OF DIFFERENTIAL EQUATION

Differential equations are one of the most important languages in science. On November 11, 1675, Gottfried Wilhelm freiherr lebinitz (1646-1716) first put the identity of the differential equation in black and white.

$$\int y\, dy = \frac{1}{2} y^2$$

What we currently refer to as the 'solution' of the differential equation was once referred to as the 'integral' of the differential equation.
In 1774, Joseph Louis Lagrange(1736-1813) coined the term "solution."
John Bernoulli is responsible for the "technique of variable separation." The solution to the differential equation was provided in a letter to Leibnitz dated May 20, 1715.

$$x^2 y' = 2y.$$

Despite the fact that Newton noted that the constant coefficient could be chosen arbitrarily and concluded that the equation had an infinite number of specific solutions, it wasn't until the middle of the 18th century that the full significance of this fact, namely that the general solution of a first order equation is dependent on an arbitrary constant, was realized. A specific equation can only be integrable in a finite form, that is, written in terms of known functions, in exceptional circumstances. In the general case, one must rely on solutions stated as infinite series with recurrence-formula determining the coefficients. In 1682, Leibniz became a associate on the new Leipzig periodical, acta eruditorum, in which he bring out his period to making six(6) page paper on the differential calculus in 1684,followed two years later (1686) by a paper under control the rudiments of the integral calculus.

1.3 BASIC DEFINITION OF DIFFERENTIAL EQUATION

1) Differential equation:- A differential equation (DE) is an equation which containing the derivatives of one or more dependent variables, with respect to one or more independent variables. is called "differential equation"

$$\frac{dy}{dx}+x=\sin(x)$$

2) Ordinary differential equation:- A differential equation is the highest order of the derivatives of the unknown function appearing the equation. is called "Ordinary differential equation"

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots\ldots\ldots\ldots\ldots$$

3) Order of differential equation:- The order of the highest order derivatives involving differential equation is called order of a differential equation.

$$\frac{d^3x}{dx^3}+3x\frac{dy}{dx}=e^y$$

4) Linear differential equation:- if the degree of the independent variable and it's derivatives is one & they don't occur linear differential equation.

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}+1=0$$

5) Non-linear differential equation:- A differential equation which not linear is called Non-linear differential equation.

$$\frac{d^3y}{dx^3}-\left(\frac{dy}{dx}\right)^2+y^2=x^2$$

6) Degree of a differential equation:- The degree of differential equation is the degree of the highest derivative which occur in it after the differential equation has been made free radical & fraction as per as the derivatives.

$$\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2-1=0 \qquad \text{Order is 2}$$

1.4 SOLUTION OF DIFFERENTIAL EQUATION

There are many types of differential equation. We can see below One by one

- **General solution:-** The solution which contains number of arbitrary independent constant equal to the order of different equation is called "General solution"
- **Particular solution:-** The solution obtained from the general solution by assign in Particular value to one or more arbitrary constant is called "particular solution"
- **Singular solution:-**The solution of differential equation which doesn't contains an Arbitrary constant is called "Singular solution"

1.5 METHOD FOR SOLVING THE DIFFERENTIAL EQUATION

1. Solution by inspection
2. Variable separable
3. Homogeneous
4. Linear differential equation

We can see below one by one

1. Solution by inspection:- if the differential equation is of the

$$f(f1(x,y))d(f1(x,y)) + \varphi(f2(x,y))d(f2(x,y)) + \cdots \ldots = 0,$$

then each term can be separately integrated.

The solution to a differential equation can be found using the inspection method.

It is accompanied by memorizing the following results.

$(i) d(x+y) = dx + d$

$(ii)\ d(xy) = xdy + ydx$

$(iii)\ d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

2. Variable separable:- If an equation can be written such that variables are separated for integration then the equation can be solved.

We get…..

$$\int f(x)dx + \int g(y)dy = c$$

1) Solve: $(x^2+1)\frac{dy}{dx} = 1$

Solution:-the given equation can be written as

$$dy = \frac{dx}{x^2+1}$$

on integration, we get

$$\int dy = \int \frac{dx}{x^2+1} + c$$

$$tan^{-1}x + c$$

This is the solution of the equation.

3. Homogenous Differential Equation:- A differential equation of form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ where f & ϕ are homogeneous is called as homogenous.

Homogenous function:-The function (x, y) is called functions. So, if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

Thus homogenous function can be written as

$$f(x,y) = x^n f\frac{y}{x}$$

Homogeneous differential equations

In first order first degree differential equation is expressed in form.

$$dy/dx = (f(x,y))/(g(x,y))$$

Example: Solve differential equation $x^2dy + y(x+y)dx = 0;\ y = 1\ when\ x = 1$

Solution:

$$x^2dy + y(x+y)dx = 0$$

$$x^2dy = -y(x+y)dx$$

$$\frac{dy}{dx} = \frac{-y(x+y)}{x^2}$$

Since each xy+y^2 & x^2 are homogeneous

Putting y=vx,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$V + x\frac{dv}{dx} = -(\frac{vx^2 + v^2x^2}{x2})$$

$$V + x\frac{dv}{dx} = -(v + v^2)$$

$$\int \frac{1}{v^2+2v} dv = -\int \frac{1}{x}$$

$$\frac{1}{2}\log\frac{|v+1-1|}{|v+1+1|} = -\log x + \log c$$

$$log\frac{|v|}{|v+2|} = 2logx = 2logc$$

$$\log\frac{|vx^2|}{|v+2|} = 2\log k$$

$$k = \frac{|vx^2|}{|y+2x|}$$

put y=1 & x=1

$$k = \frac{1}{3}$$

$$3x^2 = y + 2x$$

$$Y = \frac{2x}{3x^2-1} \quad \textit{which is a required solution}$$

4. Linear differential equation:-

Equation of form dy/dx+py= θ

A differential equation is linear if the dependent variable (y) and its derivative appear only in first degree.

The general form of linear differential equation of first order is

dy/dx+Py=Q, here P, Q are constants.

We solve such type of equation by multiplying both sides

$e^{\int dx}$, so

$$e^{\int pdx}\left(\frac{dy}{dx} + py\right) = Qe^{\int pdx}$$

$$\frac{d}{dx}\{ye^{\int pdx}\} = Qe^{\int pdx}$$

Integrating both side

$$ye^{\int pdx} = \int Qe^{\int pdx}dx + c$$

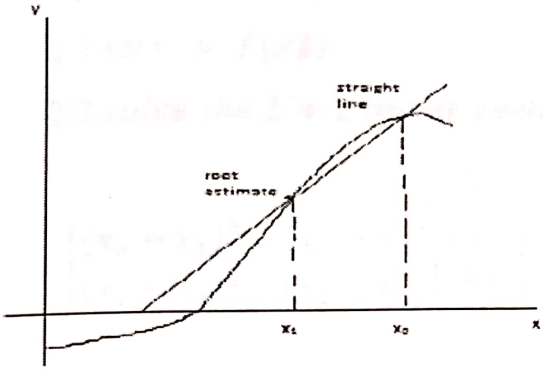
Here

$e^{\int pdx}$ is called integrating factor.

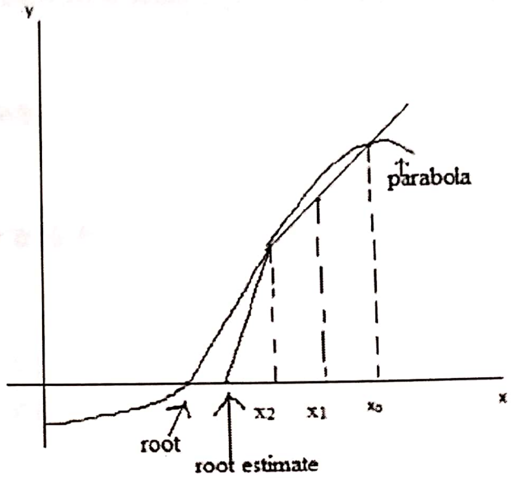
$$Y[I.F] = \int q(I.F)dx + c$$

CHAPTER: - 2

MULLER'S METHOD



(A)



(B)

2.1 INTRODUCATION

Remember that the secant estimates the root by projecting a straight to the x-axis. As illustrated in fig. a, two function values are used. Muller's method uses a similar concept, but instead projects a parabola through the three sites depicted in fig.

Muller's Approach is a numerical method for solving equations of the form f(x) = 0. It is a root seeking procedure. David E. Muller was the first to introduce it in 1956. Muller's approach is based on the secant method, which generates a line between two points on the graph of f at each iteration. Muller's technique, on the other hand, use three locations, constructs the parabola through these three points, and considers the intersection of the x-axis with the parabola to be the next best approximation.

2.2 Step To Follow In Solving The Complex Roots Using Muller's Method

1. Evaluate the function values of the initial approximations $f(x0), f(x1) \& f(x2)$

2. Compute the next approximation x_3 as follows

   2.1 set $c = f(x2)$

   2.2 $solve\ the\ 2 * 2\ linear\ system\ for\ a\ \&\ b$

$$\begin{bmatrix}(x_0 - x_1)^2 & (x_0 - x_2)\\(x_1 - x_2)^2 & (x_1 - x_2)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}f(x_0) - c\\f(x_1) - c\end{bmatrix}$$

   2.3 compute $x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$, choosing the "+" or "-" so that the denomitor is the largest in magnitude.

3. Test if $\frac{|x_3 - x_2|}{|x_3|} < \epsilon$ or if the number of iterations exceeds N. if so stop. Otherwise the step 4.

4. Ralabel x_3 as x_2 , x_2 as x_1 &x_1 as

$$x0: \begin{cases} x_2 \equiv x_3 \\ x_1 \equiv x_2 \\ x_0 \equiv x_1 \end{cases}$$

Return to step 1.

2.3 EXAMPLE ON MULLER'S METHOD

1) Use Muller's method with guesses of x_0, x_1 & x_2 are 4.5, 5.5 & 5 respectively, to Determine a root of the equation is

$$f(x)=x^3-13x-12$$

Solution:- we evaluate the function at the guesses

$x_0=4.5$ $x_1=5.5$ & $x_2=5$

$f(x)=x^3-13x-12$

$f(x_0)=f(4.5)=(4.5)^3-13(4.5)-12=20.625$

$f(x_1)=f(5.5)=(5.5)^3-13(5.5)-12=82.875$

$f(x_2)=f(5)=(5)^3-13(5)-12=48$

which can be used to calculate

$h_0=x_1-x_0=5.5-4.5=1$

$h_1=x_2-x_1=5-5.5=-0.5$

$$ə_0=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{82.875-20.625}{5.5-4.5}=\frac{62.25}{1}=62.25$$

$$ə_1=\frac{(fx_2)-f(x_1)}{x_2-x_1}=\frac{48-82.875}{5-5.5}=\frac{-34.875}{-0.5}=69.75$$

$$a=\frac{ə_1-ə_0}{h_1+h_0}=\frac{69.75-62.25}{-0.5+1}=\frac{7.5}{0.5}=15$$

$b=ah_1+ə_1=15*(-0.5)+69.75=62.25$

$c=f(x_2)=48$

The square root of the discriminate can be evaluated as

$$\sqrt{b^2}-4ac=\sqrt{(62.25)^2}-4*15*48=31.54461$$

{because $|62.25+31.54451|>|62.25-31.54451|$ a '+' ve sign is employed In the denominator}

$$Root=x2-\frac{2c}{b\pm\sqrt{b^2-4ac}}=5-\frac{2*48}{93.79451+31.54461}$$

X_3 = Root = 3.976487

We find 2nd Next iteration...

New guesses are assigned x_0 is replaced by x_1

X_1 is replaced by x_2

X_2 is replaced by x_3

Then,

$x_0=5.5$; $x_1=5$; $x_2=3.976487$

$f(x)=x^3-13x-12$

$f(x_0) = f(5.5) = (5.5)^3-13(5.5)-12 = 82.875$

$f(x_1) = f(5) = (5)^3-13(5)-12 = 48$

$f(x_2) = f(3.976487) = (3.976487)^3-13(3.976487)-12$

$= - 0.8163336$

which can be used to calculate

$h_0=x_1-x_0=5-5.5=-0.5$

$h_1=x_2-x_1=3.976487-5=-1.023513$

$$ə_0 = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{48-82.875}{5-5.5} = \frac{-34.875}{-0.5} = 69.75$$

$$ə_1 = \frac{f(x_2)-f(x_1)}{x_2-1} = \frac{-0.8163336-48}{3.976487-5} = \frac{-48.8163336}{-1.023513} = 47.6948838$$

$$a = \frac{ə_1-ə_0}{h_1+h_0} = \frac{47.6948838-69.75}{-1.023513+(-0.5)} = \frac{-22.0551162}{-1.523513} = 14.47648704$$

$b = ah_1 + d_1$

$= 14.47648704 * (1.023513) + 47.6948838$

$b = 32.87801112$

$C = f(x_2) = - 0.8163336$

The square root of the discriminate can be evaluated as

$$=> \sqrt{b^2 - 4ac}$$

$$= \sqrt{(32.87801112)^2 - 4 * 14.47648704 * (-0.8169996)}$$

$$= \sqrt{1080.963615 - 4 * 14.47648704 * (-0.8163336)}$$

$= \sqrt{1128.234181} = 33.58919739$

{Because $|32.87801112 + 33.58919739| > |32.87801112 - 33.58919739|$
a '+' ve sign is employed In the denominator}

$$Root = x2 - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$$

$$= 3.976487 - \frac{2*(-0.8163336)}{66.46720851 + 33.58919739}$$

$$= 3.976487 - \frac{(-1.6326672)}{100.0564059}$$

$$= 3.976487 - (-0.01631746799)$$

$X_3 = Root = 3.992804468$

2) Use Muller's method with guesses of x_0, x_1 & x_2 are 0.5 ,1.0 & 0.0 respectively, to Determine a root of the equation is $f(x)=3x+\sin(x)-e^x$

Solution:- we evaluate the function at the guesses

$x_0 = 0.5$; $x_1 = 1.0$ & $x_2 = 0.0$

$f(x) = 3x+\sin(x)-e^x$

$f(x_0) = f(0.5) = 3(0.5)+\sin(0.5)-e^{0.5} = -0.31999$

$f(x_1) = f(1.0) = 3(1.0)+\sin(1.0)-e^{1.0} = 0.29917$

$f(x_2) = f(0.0) = 3(0.0)^+\sin(0.0)-e^{0.0} = -1$

Which is used to calculate

$h_0=x_1-x_0=1.0-0.5=0.5$

$h_1=x_2-x_1=0.0-1.0=-1$

$$\partial_0 = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{0.29917-(-0.31999)}{1.0-0.5} = \frac{0.61916}{0.5} = 0.23832$$

$$\partial_1 = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{-1-0.29917}{0.0-1.0} = \frac{-129917}{-1} = 1.29917$$

$$a = \frac{\partial_1-\partial_0}{h_1+h_0} = \frac{1.29917-1.23832}{-1+0.5} = \frac{0.06085}{-0.5} = -0.1217$$

$b = ah_1+\partial_1 = -0.1217 * (-1)+1.29917 = 1.42087$

$c = f(x_2) = -1$

The square root of the discriminate can be evaluated as

$$\sqrt{(b)^2-4ac} = \sqrt{(1.42087)^2-4(-0.1217)(-1)}$$

$$= \sqrt{2.018871557-0.4868}$$

$$= \sqrt{1.532071551}$$

$$= 1.237768782$$

[*Because* $|1.42087+1.237768782| > |1.42087+1.237768782|$

$|2.658638782| > |0.183101218|$

A '+' sign is employed in the denominator]

$$Root = x_2 - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$$

$$= 0 - \frac{2*(-1)}{2.658638782+1.237768782}$$

$$= \frac{2*1}{3.896407564}$$

$x_3 = 0.513293326$

We find 2nd Next iteration ...

New guesses are assigned x_0 is replaced by x_1

x_1 is replaced by x_2

x_2 is replaced by x_3

Then,

$x_0 = 1.0$; $x_1 = 0.0$ & $x_2 = 0.513293326$

$f(x) = 3x+\sin(x)-e^x$

$f(x_0) = f(1.0) = 3(1.0)+\sin(1.0)-e^{1.0} = 0.29917$

$f(x_1) = f(0.0) = 3(0.0)+\sin(1.0)-e^{0.0} = -1$

$f(x_2) = f(0.513293326)$

$= 3(0.\ 513293326)+\sin(0.\ 513293326)-e^{0.0}$

$= (-0.121946)$

which is used to calculate

$h_0 = x_1 - x_0 = 0.0 - 1.0 = -1.0$

$h_1 = x_2 - x_1 = 513293326 - 0.0 = 513293326$

$$\partial_0 = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{-1-0.29917}{0.0-1.0} = \frac{-1.29917}{-1.0} = -0.70083$$

$$\partial_1 = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{-0.121946-(-1)}{0.513293326-0.0} = \frac{0.878054}{513293326} = 1.826257784$$

$$a = \frac{\partial_1 - \partial_0}{h_1+h_0} = \frac{1.826257784-(-0.70083)}{0.513293326+(-1.0)}$$

$$= \frac{2.527087784}{-0.486706674}$$

$= -5.192219296$

$b = ah_1 + \partial_1$

$= -5.192219296(0.513293326) + 1.826257784$

$= -0.838873727$

$C = f(x_2) = -0.121946$

The square root of the discriminate can be evaluated as

$= \sqrt{(b)^2 - 4ac}$

$= \sqrt{(-0.838873727)^2 - 4(-5192219296)(-0.121946)}$

$= \sqrt{0.703709129 - 2.532681497}$

$= \sqrt{-1.828972368}$

$= 1.352395049$

{Because...$|-0.838873727 + 1.352395049| > ||-0.838873727 - 1.352395049|$

$|2.191268776| > |-0.513521322|$

A '+' sign is employed in the denominator}

$$\text{Root} = x_2 - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$$

$$= 0.513293326 - \frac{2 * (-0.121946)}{2.191268776 + 1.352395049}$$

$$= 0.513293326 - \frac{(-0.243892)}{3.543663828}$$

$x_3 = 0.513293326 - (-0.068824812)$

$\text{Root} = x_3 = 0.582118138$

2.4 Advantage and disadvantage of Muller's method

- ✓ The difference between the root faster than the secant approach can be seen in the Muller's method shown above.
- ✓ Can locate fictitious roots. There's also no need to look for derivatives.
- ✓ This is where the tangential roots can be found. It'll be more precise. Then, because there are less iteration, it is faster.
- ✓ The disadvantage of Muller's is that it is much more difficult for m>2 than it is for m=1 or m = 2. It is much harder to determine the roots of a polynomial odd degree 3 or more then higher.
- ✓ There appears to be no guidance as to which of the roots of pk, m should be chosen as the next approximation of xk for the m>2.
- ✓ If the quadratic formula has a negative discriminate, you won't be able to find the root without using a software. Then, doing it by hand, can take a long time.
- ✓ This approach allows for complete view of the degenerative, native quadricepetendon, which aids in the extensor mechanism's correct function.
- ✓ It takes a long time to do by hand, so there's more chance for error. Extraneous roots can be discovered.
- ✓ Muller's method isn't necessarily more efficient than secant or other methods. Extraneous roots can be found. By hand, it's really difficult.

CHAPTER: - 3

BROYDEN'S METHOD

3.1 INTRODUCATION

We looked at the numerical method known as Newton's method in this method. We discovered that one of the method's key drawbacks was that the J(x) and its inverse had to be computed at each iteration. As a result, we want to avoid this issue. In place of the Jacobin matrix, there is a method known as the Quasi-Newton method, which burden and faires in[3] define as a method that uses an approximation matrix that is updated at each iteration. This means that Broyden's method's iterative technique is nearly identical to Newton's method's iterative procedure. The sole difference is that an approximation matrix Ai is used.

in place of J (x). As a result, the equation below is derived.

$x^{(i+1)} = x^{(i)} - A_i^{-1}F(x^{(i)})$.

This is defined as broyden's iterative procedure.

In [3], A_i is defined as

$$A_i = A_{i-1} + \frac{y_i - A_{i-1}s_i}{\|s_i\|_2^2} s_i^t$$

$y_i = F(x^{(i)}) - F(x^{(i-1)})$ and $s_i = x^{(i)} - x^{(i-1)}$. However, in broyden's method it involves that Computation A_i^{-1}, not A_i,

3.2 Steps to follow in solving for Broyden's method

Step 1:

Let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots\ldots\ldots\ldots x_n^{(0)})$ be the initial vector given.

Step 2:

Calculate $F(x^{(0)})$.

Step 3:

In this step we compute A_0^{-1}. Because we do not have enough information to Computer A_0 directly. Broyden's method permits us to let $A_0 = J(x^{(0)})$, which Implies that $A_0^{-1} = J(x^{(0)})^{-1}$.

Step 4:

Calculate $X^{(1)} = x^{(0)} - A_0^{-1}F(x^{(0)})$.

Step 5:

Calculate $F(x^{(1)})$.

Step 6:

Take $F(x^{(0)})$ & $F(x^{(1)})$ & calculate $y_1 = F(x^{(1)}) - F(x^{(0)})$. Next, take the first two iterations Of $x^{(i)}$ and calculate $s_1 = x^{(1)} - x^{(0)}$.

Step 7:

Calculate $s_1^t A_0^{-1} y_1$.

Step 8:

Compute $A_1^{-1} = A_0^{-1} + \frac{1}{s_1^t A_0^{-1} y_1} [(s_1 - A_0^{-1} Y_1) s_1^t A_0^{-1}]$

Step 9:

Take A_1^{-1} that we found in step 8, and calculate

$x^{(2)} = x^{(1)} - A_1^{-1}F(x^{(1)})$.

Step 10:

Repeat the process until e converge to $\bar{x}$, i.e when $x^{(i)} = x^{(i+1)} = \bar{x}$

this will indicate that we have reached the solution of the system.

3.3 PROBLEMS ON BROYDEN'S METHOD

Example1: $g = \begin{bmatrix} x^2+y^2+z^2-3 \\ x^2+y^2-z-1 \\ x+y+z-3 \end{bmatrix}$

Solution:- $g = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_3}{\partial z} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2 & 2y & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$D_0 = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solve $D_0 d^{(0)} = -g(x^{(0)})$.

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} d^{(0)} = -\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad d^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$x^{(1)} = x^{(0)} + d^{(0)} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$y^{(0)} = g(x^{(1)}) - g(x^{(1)}) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

$$y^{(0)} - D_0 d^{(0)} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$D_1 = D_0 + \frac{(y^0 - D_0 d^0) * d^0}{d^0 * d^0}$$

Since $d^{(0)} * d^{(0)} = \frac{1}{2}$

$$\Rightarrow \quad D_1 = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} & 2 \\ \frac{5}{2} & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$D_1 d^{(1)} = -g(x^{(1)})$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & \frac{1}{2} & 2 \\ \frac{5}{2} & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{bmatrix} d^{(1)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad X^{(2)} = \begin{bmatrix} \frac{5}{2} \\ \frac{3}{4} \\ 1 \end{bmatrix}$$

Nice properties of broyden's

1. $d(D_{k+1}-D_k) \le (D, D_k)$

If $Dd^{(k)} = y^{(y)}$, where

$$d(A, B) = \max \{ \| Ax-Bx \| : \| x \| \le 1\}$$

$$2. \lim_{k\to\infty} \frac{(D_k - g(x^{(k)})(x^k - x)^*}{x^k - x^*} = 0$$

Where $g(x^*) = 0$, i.e $x^{(k)} \to x^*$ and $D_k(x^{(k)}-x^*)$ acks likeg $(x^*)(x^{(k)}-x^*)$.

Drawback: D_k is not SPD.

Example 2) solving the system of equations.

$$\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$$

Solution:- We determined that taking the derivatives of the linear function was too difficult for us at this point. As a result, we'll make an educated guess.

$(y_0)=(0,0)$ and

$D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ our functions g is $g(x,Y) = \begin{bmatrix} x + y - 2 \\ x - y \end{bmatrix}$

Our 1st step sets.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - D_0^{-1} g(x_0, y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

So the step we took is $b^{(0)} = (2, 0)$, and $g(x_1, y_1) = (0, 2)$ using the update formula

$$D_1 = D_0 + \frac{g(x_1, y_1).(b^0)^T}{b^0.b^0}$$

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(Newton's method, or even Broyden's method with an accurate D0, would have gotten the answer in this first step, but we'll need a bit more work.)

Our second step sets.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

So the step we took is $b^{(1)}=(0, -2)$ & $g(x_2, y_2)=(-2,4)$. We do another rank- one Update to get

$$D_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} -2 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

D_2 becomes the correct Jacobian matrix of the linear function g now that there is more information at more places. Our linear approximation to g will equal g in the following step, and (x_3, y_3) will be the correct solution (1, 1).

In general, it is not necessarily true that the matrix D_0 will always resemble $\nabla g(x^k)$ after many steps after an incorrect guess of D_0. After a sufficient number of steps, the matrix D_k will frequently precisely tell us what $\nabla g(x^k)$ does in the relevant directions: the ones we actually need to use.

3.4 Advantages and disadvantages of broyden's method

Broyden's approach offers several benefits, the most prominent of which is the reduction in computations. It's the inverse of Ai-1, the approximation matrix that may be derived straight from the previous iteration. Ai-1-1 reduces the amount of computations required for this method from Needed to Needed when compared to Newton's approach.

This quasi-newton technique does not converge to the quadratically, which is seen as a drawback.

This could imply that further iterations are needed to reach the desired result. In comparison to Newton's technique, which necessitates a large number of iterations.

Another issue with Broyden's approach is that it is not self-correcting, as burden. And faires discuss in [3]. This suggests that, unlike Newton's method, it is ineffective.

This could indicate that further iterations are required to achieve the desired outcome. In comparison to Newton's method, which requires a significant number of iterations.

3.5 APPLICATION

- ✓ Differential equations are a broad subject of study in pure and applied mathematics, physics, and engineering. All of these areas are concerned with the characteristics of various forms of differential equations.

- ✓ Pure mathematics is concerned with the existence and uniqueness of solutions, whereas applied mathematics is concerned with the justification of approximation methods.

- ✓ Differential equations are used to simulate nearly every physical, technical, or biological activity, ranging from celestial motion to bridge design to neuron connections.

- ✓ Differential equations used to solve real-world issues aren't always directly solvable, i.e. they don't always have closed form solutions. Numerical approaches can be used to approximate solutions instead.

- ✓ Many fundamental laws of physics and chemistry can be formulated as differential equations

- ✓ Differential equations are used to model the behaviour of complex systems in biology and economics.

- ✓ The importance of the issue is evidenced by the number of differential equations that have been named in many scientific fields. See the list of named differential equations for more information.

- ✓ Differential equations can be solved with some CAS software. These CAS programmes and their commands are noteworthy.

- ✓ Some CAS software can solve the differential equations. These CAS software and their commands are worth mentioning:

Mathematica:[16] DSolve[]

Maple:[15] dsolve [17]

desolve SageMath ()

desolve(y'=k*y,y) Xcas:[18]

3.6 CONCLUSION

It is safe to say that numerical methods are the most important aspect of mathematics, based on the information presented in this work. They are a useful tool since they can solve nonlinear algebraic problems in one variable as well as a system of nonlinear algebraic equations.

Muller's approach is an iterative algorithm for determining function roots. These functions can be real or complex valued, and they aren't limited to any one function subspace. Muller's approach combines the speed of Newton's method with the computational ease of the secant method, as it does not require the computing of derivatives. As a result, it performs admirably, and convergence to the method is dependent on the initial points chosen.

Broyden method is a method for solving a system of nonlinear equations that aims to improve Newton's method in terms of storage and approximation to the Jacobian. Broyden method is a generalization of the secant method to many dimensions.

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